

# ABOUT INTEGRABILITY OF ALMOST COMPLEX STRUCTURES ON STRICTLY NEARLY KÄHLER 6-MANIFOLDS

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ABSTRACT. We show that any almost complex structure, positively tamed with  $\omega$  on nearly Kähler 6-manifold  $(M, g, J, \omega)$  is not integrable.

**1. Introduction.** Let  $(M, g, J, \omega)$  is an almost Hermitian manifold, where  $g$  is a Riemannian metric,  $J$  an almost complex structure compatible with  $g$  and  $\omega$  is Kähler form,  $\omega(X, Y) = g(JX, Y)$ . Nearly Kähler manifold is an almost Hermitian manifold  $(M, g, J, \omega)$  with the property that  $(\nabla_X J)X = 0$  for all tangent vectors  $X$ , where  $\nabla$  denotes the Levi-Civita connection of  $g$ . If  $\nabla_X J \neq 0$  for any non-zero vector field  $X$ ,  $(M, g, J, \omega)$  is called strictly nearly Kähler. Nearly Kähler geometry comes from the concept of weak holonomy introduced by A. Gray in 1971 [1], this geometry corresponds to weak holonomy  $U(n)$ . The class of nearly Kähler manifolds appears naturally as one of the sixteen classes of almost Hermitian manifolds described by the Gray-Hervella classification [2]. Nagy P.-A. has proved [3] that every compact simply connected nearly Kähler manifold  $M$  is isometric to a Riemannian product  $M_1 \times \dots \times M_k$ , such that for each  $i$ ,  $M_i$  is a nearly Kähler manifold belonging to the following list: Kähler manifolds, naturally reductive 3-symmetric spaces, twistor spaces over compact quaternion-Kähler manifolds with positive scalar curvature, and nearly Kähler 6-manifold. This is one of the reasons of interest to the nearly Kähler 6-manifolds. In case of dimension 6 there exists several equivalent conditions defining a strictly nearly Kähler structures  $(g, J, \omega)$  on  $M$ . For example, conditions

- (i)  $(\nabla_X J)Y$  is skew-symmetric with respect to  $X, Y$  and non-zero;
- (ii) The form  $\nabla\omega$  is non-zero, totally skew-symmetric and  $\nabla_X\omega = \frac{1}{3}\iota_X d\omega$ ,  $\forall X \in \Gamma(TM)$ ;
- (iii) The structure group of  $M$  admits a reduction to  $SU(3)$ , that is, there is  $(3,0)$ -form  $\Omega$  with  $|\Omega| = 1$ , and

$$d\omega = 3\lambda \operatorname{Re} \Omega, \quad d \operatorname{Im} \Omega = -2\lambda\omega^2$$

where  $\lambda$  is a non-zero real constant, are equivalent and define strictly nearly Kähler manifold (see [4]).

By (iii), for strictly nearly Kähler structure  $(g, J, \omega)$  the  $d\omega \neq 0$  and of type  $(3,0)+(0,3)$ , so  $J$  is not integrable. It is natural to ask about the possibility of

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"neighborhood" of nearly Kähler structure  $J$  with integrable one. It is known [5] that the almost complex structure, underlying a strictly nearly Kähler 6-manifold cannot be compatible with any symplectic form, even locally. In this article we give answer on question: do there exist integrable almost complex structure, compatible with  $\omega$ ?

**2. The main result.** Denote by:

$A^+$  – the space of all positive oriented almost complex structures (a.c.s.) on  $M$ .

$A_\omega^+$  – the set of all almost complex structures on  $M$ , tamed by form  $\omega$ .

$$A_\omega^+ = \{I \in A^+ : \omega(IX, IY) = \omega(X, Y), \forall X, Y \in \Gamma(TM); \omega(X, IX) > 0, \forall X \neq 0\}$$

$AO_g^+$  – the set of all  $g$ -orthogonal positive oriented almost complex structures.

$$AO_g^+ = \{I \in A^+ : g(IX, IY) = g(X, Y), \forall X, Y \in \Gamma(TM)\}$$

**Lemma.** For arbitrary a.c.s.  $I \in A_\omega^+$  endomorphism  $J + I$  is non-degenerate.

*Proof.* Suppose that there exists the a.c.s.  $I \in A_\omega^+$ , for which  $J + I$  is degenerate. Then  $(J + I)_x(X) = 0$  for some  $x \in M$ ,  $X \in \Gamma(TM)$ . Therefore  $\omega_x(X, JX) = -\omega(X, IX)$ , what contradicts to positivity of  $\omega(X, JX)$  and  $\omega(X, IX)$ . Lemma is proved.

As a consequence of lemma for each a.c.s.  $I \in A_\omega^+$  one can find endomorphism:

$$K = (J + I)^{-1}(I - J) = (1 - IJ)^{-1}(1 + IJ)$$

It is easy to show, that for  $K$ :

1.  $KJ = -JK$ ;
2.  $g(KX, Y) = g(X, KY)$ ,  $\forall X, Y \in \Gamma(TM)$ ;
3.  $1 - K^2 > 0$

Properties 1 and 2 give additionally that  $\omega(KX, Y) = -\omega(X, KY)$ . On the other hand, any endomorphism  $K$ , with properties 1-3 defines a.c.s.  $I = (1 - K)J(1 - K)^{-1} \in A_\omega^+$ .

**Theorem.** Any almost complex structure  $I \in A_\omega^+$  is non-integrable.

*Proof.* Let  $X \in \Gamma(TM)$  – is field of eigenvectors of  $1 - K^2$  with eigenvalue  $\lambda$ . Then  $(1 - K^2)JX = J(1 - K^2)X = \lambda JX$ , and  $(1 - K^2)(X - iJX) = \lambda(X - iJX)$ . So one can define  $g$ -orthogonal basis  $\nu^1, \nu^2, \nu^3$  in the space of  $(1,0)$ -forms, which are eigenvectors of  $1 - K^2$  in  $T^*M \otimes \mathbb{C}$ .

Form  $\omega(\nu^k, \nu^l) = g(J\nu^k, \nu^l) = ig(\nu^k, \nu^l) = 0$ , for  $k, l = 1, 2, 3$  so  $\omega$  is of type  $(1,1)$  in the basis  $(\nu, \bar{\nu})$ .

Let define forms  $\theta^k = (1 - K)\nu^k$ ,  $k = 1, 2, 3$  in  $T^*M \otimes \mathbb{C}$ . One can see, that  $I\theta^k = (1 - K)J(1 - K)^{-1}(1 - K)\nu^k = i(1 - K)\nu^k = i\theta^k$ . Therefore,  $\theta^1, \theta^2, \theta^3$  – are linear independent  $(1,0)$ -forms, with respect of  $I$ .

Find value of form  $\omega$  on  $\theta^k, \theta^l$  for arbitrary  $k, l = 1, 2, 3$ :

$$\omega(\theta^k, \theta^l) = \omega((1 - K)\nu^k, (1 - K)\nu^l) = \omega(\nu^k, (1 - K^2)\nu^l) = \lambda\omega(\nu^k, \nu^l) = 0.$$

Therefore form  $\omega$  has type  $(1,1)$  in basis  $(\theta, \bar{\theta})$ .

Find  $d\omega^{(3,0)}$  in  $(\theta, \bar{\theta})$ :

$$\begin{aligned} d\omega &= \nu^1 \wedge \nu^2 \wedge \nu^3 + \bar{\nu}^1 \wedge \bar{\nu}^2 \wedge \bar{\nu}^3 = \\ &= (1-K)^{-1}\theta^1 \wedge (1-K)^{-1}\theta^2 \wedge (1-K)^{-1}\theta^3 + \overline{(1-K)^{-1}\theta^1} \wedge \overline{(1-K)^{-1}\theta^2} \wedge \overline{(1-K)^{-1}\theta^3} \end{aligned}$$

Let's calculate the  $d\omega^{(3,0)}$ . In local frame  $(\nu, \bar{\nu})$  the matrix of operator  $1-K$  is  $\begin{pmatrix} 1 & \bar{V} \\ V & 1 \end{pmatrix}$ , where  $V^T = V$ ,  $1 - V\bar{V} > 0$ . It is easy to check, that  $\begin{pmatrix} 1 & \bar{V} \\ V & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Lambda^{-1} & -\bar{V}\Lambda^{-1} \\ -V\Lambda^{-1} & \Lambda^{-1} \end{pmatrix}$ , where  $\Lambda$  - is diagonal matrix of eigenvalues of  $1-K^2$ , corresponding to  $(\nu^1, \nu^2, \nu^3)$ . Therefore

$$d\omega^{(3,0)} = \frac{1}{\sqrt{\det(1-K^2)}} \theta^1 \wedge \theta^2 \wedge \theta^3 \neq 0$$

As  $\omega$  is of type  $(1,1)$ , then  $d\omega^{(3,0)} \neq 0$  gives  $d^{2,-1} \neq 0$  for  $I$ , and shows non-integrability of this structure. Theorem is proved.

*Remark.* The proof of the above theorem shows that really we needn't in nearly Kähler structure, we just use, that  $d\omega^{(3,0)+(0,3)} \neq 0$ , and  $d\omega^{(1,2)+(2,1)} = 0$ .

**3. Almost complex structures on  $S^6$ .** It is known that  $S^6$  admits the set of nearly Kähler structures. All of them are orthogonal with respect to round metric  $g_0$ ,  $G_2$  invariant and form the space  $\mathbb{R}P^7 = SO(7)/G_2$ . All  $g_0$  orthogonal almost complex structures on  $S^6$  are not integrable [6]. For Riemannian manifold  $(M, g_0)$  admitting almost complex structure [7] the space  $A^+$  is a smooth locally trivial fiber bundle over the space  $AO_{g_0}^+$ , with fiber  $A_{\omega_J}^+$  over  $J \in AO_{g_0}^+$ , where  $\omega_J(X, Y) = g_0(JX, Y)$ .

So the above theorem let us to enlarge the number of non integrable almost complex structures by the structures in fibers over the nearly Kähler ones.

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